NOTES FOR SECTION 7.1

Question: How to find all the eigenvalues and eigenvectors of an $n \times n$ matrix A?

Step 1. Find the characteristic polynomial of A

$$p(\lambda) = |\lambda I_n - A|.$$

By the fundamental theorem of algebra,

$$p(\lambda) = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_r)^{k_r},$$

where $\lambda_1, \cdots, \lambda_r$ are different constants. Then $p(\lambda) = 0$ has roots

$$\lambda_1, \cdots, \lambda_r.$$

They are all the eigenvalues of A.

Step 2. The space of eigenvectors corresponding to the eigenvalue λ_i is called the eigenspace of λ_i, denoted by E_i.

Then

$$E_i = N(\lambda_i I_n - A),$$

that is the eigenvectors corresponding to λ_i are the solutions to

$$(\lambda_i I_n - A)v = 0.$$

Two important facts:

1. Assume that $\dim E_i = \dim N(\lambda_i I_n - A) = m_i$. Then $1 \leq m_i \leq k_i$.
2. Assume that $\{v_i, \cdots, v_{i,m_i}\}$ is the basis for $E_i = N(\lambda_i I_n - A)$. Then

$$\cup_{i=1}^{r} \{v_i, \cdots, v_{i,m_i}\} \quad \text{are linearly independent.}$$

Diagonalization: Assume that A is an $n \times n$ matrix, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all the eigenvalues of A (counting multiplicity, which means some of them are the same). Suppose that v_i is an eigenvector with respect to λ_i. Moreover,

$$\{v_1, \cdots, v_n\} \quad \text{form a basis for } \mathbb{R}^n.$$

Let $P = [v_1 \cdots v_n]$ and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then

$$AP = [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n] = PD.$$

Hence

$$P^{-1}AP = D.$$

Definition 1. *Two $n \times n$ matrices A, B are call similar if there is an invertible $n \times n$ matrix P such that*

$$P^{-1}AP = B.$$

We usually denote it by $A \sim B$.